GENERATING CONJECTURE AND SOME EINSTEIN-MAXWELL FIELD OF HIGH SYMMETRY

M. ŠTEFANÍK, J. HORSKÝ
Department of Theoretical Physics and Astrophysics, Masaryk University, Kotlářska 2, 611 37 Brno, Czech Republic

For stationary cylindrically symmetric solutions of the Einstein-Maxwell equation we have shown that the „charged" solutions of McCrea, Chitre et al. (CGN), Van den Bergh and Wils (VW) can be obtained from the seed metrics using generating conjecture. The McCrea „charged" solution has as a seed vacuum metric the Van Stockum solution with a Killing vector $(0, 0, 1, 0)$. The CGN „charged" solution and the VW „charged" solution have the static seed metrics connected by the complex substitution $t \to iz$, $z \to it$ and the Killing vector which is a simple linear combination of ∂_φ and ∂_t Killing vectors (VW), respectively ∂_φ and ∂_z Killing vectors (CGN).

1 Introduction

There exist different methods to find the solutions of the Einstein–Maxwell equations [1]. Some high symmetry solutions of the Einstein equations have direct astrophysical or cosmological interpretation [2]. In [3], a new conjecture was formulated which says: „A test electromagnetic field having a potential proportional to the Killing vector of a seed vacuum gravitational field (up to a constant factor), is simultaneously an exact electromagnetic potencial of the self consistent Einstein–Maxwell problem where the metric is stationary or static and it reduces to the mentioned seed metric when the parameter K characterizing the charge of the electromagnetic field source vanishes."
For exact seed solution of the Einstein's equations we write

$$ds^2_{seed} = g_{ik} dx^i dx^k \,, \qquad i, k = 0, ..., 3. \tag{1}$$

Let this metric has some symmetry expressed by the Killing vector field ξ. Acording to the conjecture the electromagnetic four-potential A_i is

$$A_i = K\xi_i = K\left(g_{ik}\xi^k\right) \,. \tag{2}$$

For the „charged" solution of the Einstein–Maxwell equations we shall write

$$ds^2_{new} = \bar{g}_{ik} dx^i dx^k \,. \tag{3}$$

From (3) we obtain the seed metric (1) as a limit for $K \to 0$.
Let the „charged" solution (3) and the electromagnetic four-potential (2) are known, we shall try to obtain the seed metric (1) and the Killing vector field ξ.

2 Calculations

Let us consider, first, the McCrea solution [4]. The nonzero „charged" components of it are

$$\bar{g}_{00} = 4q^2 r^2 + C_1 r \ln(kr) \ , \qquad \bar{g}_{02} = -r \ , \qquad \bar{g}_{11} = \bar{g}_{33} = -\frac{1}{\sqrt{r}} \tag{4}$$

and the nonzero components of the seed metric ($q \to 0$ in (4)) are

$$g_{00} = C_1 r \ln(kr) \ , \qquad g_{02} = \bar{g}_{02} \ , \qquad g_{11} = g_{33} = \bar{g}_{11} = \bar{g}_{33} \ . \tag{5}$$

The electromagnetic four-potential of the McCrea's solution has the form

$$A_i dx^i = -qr \ dt \ . \tag{6}$$

From (2) we get

$$-qr = K\left(g_{02}\xi^2\right) \ , \qquad g_{02} = -r \ . \tag{7}$$

Taking $K = q$ we have the Killing vector

$$\xi^i \equiv (0, 0, 1, 0) \tag{8}$$

which is just the Killing vector ∂_φ. The „charged" solution [3] was obtained by the conjecture for the Killing vector (8) using the seed metric (5). This solution can be transformated to the McCrea's solution [5]. Therefore, we can formulate the following theorem: „The „charged" McCrea's solution has as the seed metric the Van Stockum's solution with the Killing vector field (8)."

As the second example we will consider the „charged" solution of CGN [4]. The nonzero components of (3) in the CGN's solution are

$$\bar{g}_{00} = r^{-4/9} \exp a^2 r^{2/3} \ , \bar{g}_{02} = a r^{4/3} \ , \bar{g}_{22} = -\left(r^{4/3} + a^2 r^2\right) \ , \bar{g}_{11} = -r^{4/9} \exp a^2 r^{2/3} \ , \bar{g}_{33} = -r^{2/3}. \tag{9}$$

„Charging" parameter of this solution is denoted in [4] as a, for the nonzero seed metric coefficients we get

$$g_{00} = r^{-4/9} \ , \qquad g_{22} = -r^{4/3} \ , \qquad g_{11} = -r^{-4/9} \ , \qquad g_{33} = -r^{2/3} \ . \tag{10}$$

The electromagnetic four-potential is

$$A_i dx^i = -\frac{a}{\sqrt{2}} r^{2/3} dz + \frac{a^2}{\sqrt{8}} r^{4/3} d\varphi \ . \tag{11}$$

If we put $K = a$ in (2) then the Killing vector of the seed metric (1) is

$$\xi \equiv \left(0, 0, -\frac{a}{2}\xi^3, \xi^3\right) \ , \qquad \xi^3 = \frac{1}{\sqrt{2}} \ . \tag{12}$$

It is the linear combination of ∂_φ and ∂_z Killing vectors.

Finally, let us consider the „charged" solution of VW [4]. The nonzero metric coefficients of (3) are

$$\bar{g}_{00} = r^{2/3} \ , \quad \bar{g}_{02} = -a r^{4/3} \ , \quad \bar{g}_{22} = -\left(r^{4/3} - a^2 r^2\right) \ , \quad \bar{g}_{11} = \bar{g}_{33} = -r^{-4/9} \exp -a^2 r^{2/3} \tag{13}$$

in the seed metric (1) the following „g's" are nonzero

$$g_{00} = r^{2/3} \ , \qquad g_{22} = -r^{4/3} \ , \qquad g_{11} = g_{33} = -r^{-4/9} \ . \tag{14}$$

The line element (13) has the electromagnetic four-potential in the form

$$A_i dx^i = -\frac{a}{\sqrt{2}} r^{2/3} dt + \frac{a^2}{\sqrt{8}} r^{4/3} d\varphi \tag{15}$$

and it gives us the Killing vector

$$\xi \equiv \left(\xi^0, 0, \frac{a}{2}\xi^0, 0\right) \ , \qquad \xi^0 = -\frac{1}{\sqrt{2}} \ . \tag{16}$$

This vector is the linear combination of ∂_φ and ∂_t Killing vectors.

The components of (14) are coincident with (10) by the complex substitution

$$t \to iz \ , \qquad z \to it \ . \tag{17}$$

Both „charged" solutions (13) and (9) have the similar electromagnetic four-potential. Therefore, we can formulate the following theorem: „The „charged" metrics a) CGN and b) VW can be obtained by the conjecture (2) taking a) the static seed metric (10) and the Killing vector field (12), respectively b) the static seed spacetime (14) and the Killing vector (16)."

3 Conclusion

The generating conjecture formulated in [3] can be used in both directions. If we know the „charged" static or stationary symmetric solution and the electromagnetic four-potential then we can obtain the seed metric with the Killing vector field belonging to this seed metric. Two theorem were established. The first one says: If we take the case II of the Van Stockum vacuum solution [6] and the Killing vector (8) then we obtain the „charged" solution of McCrea. The second theorem says: If we consider a) the static seed vacuum metric (10) and the Killing vector field (12), respectively b) (14) and (16) then we can obtain a) the CGN „charged" solution, respectively b) the VW „charged" solution.

The seed spacetimes (14) and (10) are connected by the complex substitution (17). Such considerations can be used in other cases as well.

References

[1] Kramer D., Stephani H., MacCallum M.A.H., and Herlt E.: *Exact solutions of Einstein's field equations.* VEB DAW, Berlin, 1980.

[2] Bičák J., Lynden-Bell D., and Pichon C.: Mon. Not. R. Astron. Soc. **265** (1993) 126.

[3] Horský J., and Mitskievitch N.V.: Czech. J. Phys. B **39** (1989) 957.

[4] MacCallum M.A.H.: J. Phys. A: Math. Gen. **16** (1983) 3853.

[5] Štefaník M., and Horský J.: Scripta Masaryk University **24-26** (1996) 39.

[6] Bonnor W. B.: J. Phys. A: Math. Gen. **13** (1980) 2121.